Çankaya University
Department of Mathematics
2020 - 2021 Spring Semester

MATH 410 - Special Functions

Final Examination

1) Show that $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^9 x \, dx = \frac{1}{560}$

Hint: In the Beta function definition $B(a,b) = \displaystyle\int_0^1 t^{a-1} \left(1-t\right)^{b-1} dt$ use the substitution $t = \sin^2 x$.

2) Show that $\displaystyle\int \operatorname{erf}(x) \, dx = x \operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + c$

Hint: Use integration by parts.

3) Show that $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

4) Solve the differential equation $xy'' + (1-x)y' + 3y = 0$ using power series method. Find a polynomial solution.

5) Show that $\ln \dfrac{1+x}{1-x} = 2xF\left(\dfrac{1}{2}, 1, \dfrac{3}{2}; x^2\right)$ where F is the hypergeometric function.

ANSWERS

1) We know that $B(a,b) = \displaystyle\int_0^1 t^{a-1}\left(1-t\right)^{b-1} dt$. Let's use the substitution $t = \sin^2 x$.

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}\left(1-t\right)^{b-1} dt \\ &= \int_0^{\frac{\pi}{2}} \sin^{2a-2} x\left(1-\sin^2 x\right)^{b-1} 2\sin x\cos x dx \\ &= \int_0^{\frac{\pi}{2}} 2\sin^{2a-1} x\,\cos^{2b-1} x\,dx \end{aligned}$$

We also know that $B(a,b) = \dfrac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$, so:

$$\int_0^{\frac{\pi}{2}} \sin^{2a-1} x\,\cos^{2b-1} x\,dx = \frac{1}{2}\cdot\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Now insert $a = 4,\ b = 5$ to obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 2\sin^7 x\,\cos^9 x\,dx &= \frac{1}{2}\cdot\frac{\Gamma(4)\,\Gamma(5)}{\Gamma(9)} \\ &= \frac{1}{2}\cdot\frac{3!\cdot 4!}{8!} \\ &= \frac{1}{560} \end{aligned}$$

2) Let's use integration by parts to evaluate $\displaystyle\int \operatorname{erf}(x)\,dx$ where $u = \operatorname{erf}(x)$ and $dv = dx$.

We know that $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$ therefore $du = \dfrac{2}{\sqrt{\pi}}\,e^{-x^2}\,dx, \quad v = x.$

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int \operatorname{erf}(x)\,dx &= x\operatorname{erf}(x) - \int x\frac{2}{\sqrt{\pi}}\,e^{-x^2}\,dx \\ &= x\operatorname{erf}(x) - \frac{1}{\sqrt{\pi}}\int e^{-x^2}\,2xdx \\ &= x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\int e^{-x^2}\,(-2x)dx \\ &= x\operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + c \end{aligned}$$

3) Let's use the explicit summation formula

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}$$

$$J_{-\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{1}{2})} \left(\frac{x}{2}\right)^{2k-\frac{1}{2}}$$

We know that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(k+1) = k\Gamma(k)$. Using these, we obtain:

$$\begin{aligned}
\Gamma\left(k+\tfrac{1}{2}\right) &= \left(k-\tfrac{1}{2}\right)\Gamma\left(k-\tfrac{1}{2}\right) \\
&= \left(k-\tfrac{1}{2}\right)\left(k-\tfrac{3}{2}\right)\Gamma\left(k-\tfrac{3}{2}\right) \\
&\vdots \\
&= \left(k-\tfrac{1}{2}\right)\left(k-\tfrac{3}{2}\right)\cdots\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right) \\
&= \frac{(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}{2^k}\sqrt{\pi}
\end{aligned}$$

Multiply and divide by $(2k)(2k-2)\cdots 6\cdot 4\cdot 2$ to obtain:

$$= \frac{(2k)!}{2^{2k}\,k!}\sqrt{\pi}$$

Insert this in the explicit formula above:

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k\,2^{2k}}{(2k)!\,\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k-\frac{1}{2}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k\,2^{2k}}{(2k)!\,\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k} \sqrt{\frac{2}{x}} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k\,2^{2k}}{(2k)!} \left(\frac{x}{2}\right)^{2k} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{2k}}{(2k)!} \\
&= \sqrt{\frac{2}{\pi x}} \cos x
\end{aligned}$$

4) Let's insert the series $y = \sum_{k=0}^{\infty} a_k x^k$ in the equation $xy'' + (1-x)y' + 3y = 0$:

$$x \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + (1-x) \sum_{k=1}^{\infty} k a_k x^{k-1} + 3 \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-1} + \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 3 a_k x^k = 0$$

Replace k by $k+1$ in the first and second summations.

$$\sum_{k=1}^{\infty} (k+1) k a_{k+1} x^k + \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 3 a_k x^k = 0$$

$$a_1 + 3a_0 + \sum_{k=1}^{\infty} \Big[(k+1)^2 a_{k+1} - (k-3) a_k \Big] x^k = 0$$

$$\Rightarrow \quad a_1 = -3a_0, \qquad a_{k+1} = \frac{(k-3)}{(k+1)^2}\, a_k \quad \text{for} \quad k = 1, 2, 3, \ldots$$

$$a_2 = -\frac{1}{2}\, a_1 = \frac{3}{2}\, a_0, \qquad a_3 = -\frac{1}{9}\, a_2 = -\frac{1}{6}\, a_0, \qquad a_4 = 0,\ a_5 = 0, \ldots$$

Let's choose $a_0 = 6$. In that case, $a_1 = -18, a_2 = 9, a_3 = -1$.

$$y(x) = -x^3 + 9x^2 - 18x + 6$$

This is 6 times $L_3(x)$. (Laguerre polynomial of order 3.)

5) Let's use Taylor series expansions:

$$\begin{aligned} \ln(1+x) - \ln(1-x) &= \Big(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \Big) - \Big(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \Big) \\ &= 2x + \frac{2}{3}\, x^3 + \frac{2}{5}\, x^5 + \cdots = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \end{aligned}$$

$$\begin{aligned} 2xF\left(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2\right) &= 2x \left(1 + \frac{\frac{1}{2} \cdot 1}{\frac{3}{2}}\, x^2 + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot 2!}{\frac{3}{2} \cdot \frac{5}{2}}\, \frac{x^4}{2!} + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot 3!}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}\, \frac{x^6}{3!} + \cdots \right) \\ &= 2x \left(1 + \frac{1/2}{3/2}\, x^2 + \frac{1/2}{5/2}\, x^4 + \frac{1/2}{7/2}\, x^6 + \cdots \right) \\ &= 2 \left(x + \frac{1}{3}\, x^3 + \frac{1}{5}\, x^5 + \frac{1}{7}\, x^7 + \cdots \right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \end{aligned}$$