Çankaya University
Department of Mathematics
2020 - 2021 Spring Semester

MATH 410 - Special Functions

First Midterm Examination

1) Show that $\displaystyle\int_0^\infty \sqrt{x}e^{-x^3}\,dx = \frac{\sqrt{\pi}}{3}$.

2) Evaluate $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx$.

3) Show that $\displaystyle\int_0^x t^2 e^{-4t^2}\,dt = \frac{\sqrt{\pi}}{32}\,\mathrm{erf}(2x) - \frac{x}{8}\,e^{-4x^2}$.

4) We are given a set of functions $F_n(x)$ that satisfies the recurrence relations

$$\begin{aligned}\frac{2n}{x}\,F_n(x) &= F_{n-1}(x) + F_{n+1}(x)\\ 2\,F_n'(x) &= F_{n-1}(x) - F_{n+1}(x)\end{aligned}$$

Find a second order ODE whose solution is $F_n(x)$.

5) Show that

$$e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$$

$$e^{-x} = I_0(x) + 2\sum_{n=1}^{\infty} (-1)^n I_n(x)$$

where $I_n(x)$ is the modified Bessel function of order n.

ANSWERS

1) Use substitution $u = x^3 \quad \Rightarrow \quad du = 3x^2 dx.$

$$x = u^{1/3} \quad \Rightarrow \quad dx = \frac{du}{3u^{2/3}}$$

$$\sqrt{x} = u^{1/6}$$

$$\begin{aligned}
\int_0^\infty \sqrt{x} e^{-x^3}\, dx &= \int_0^\infty u^{1/6}\, e^{-u}\, \frac{du}{3u^{2/3}} \\
&= \frac{1}{3} \int_0^\infty u^{-3/6}\, e^{-u}\, du \\
&= \frac{1}{3} \int_0^\infty u^{1/2-1}\, e^{-u}\, du
\end{aligned}$$

Using the definition of gamma function as $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1} e^{-x}\, dx$ we obtain

$$\begin{aligned}
\int_0^\infty \sqrt{x} e^{-x^3}\, dx &= \frac{\Gamma\left(\frac{1}{2}\right)}{3} \\
&= \frac{\sqrt{\pi}}{3}
\end{aligned}$$

2) Beta function definition is: $B(x, y) = \displaystyle\int_0^1 t^{x-1} (1-t)^{y-1}\, dt$

$$\begin{aligned}
\int_0^1 \sqrt{t(1-t)}\, dt &= \int_0^1 t^{1/2}\, (1-t)^{1/2}\, dt \\
&= \int_0^1 t^{3/2-1}\, (1-t)^{3/2-1}\, dt \\
&= B\left(\tfrac{3}{2}, \tfrac{3}{2}\right)
\end{aligned}$$

Using $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain

$$\begin{aligned}
\int_0^1 \sqrt{t(1-t)}\, dt &= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} \\
&= \frac{\left[\frac{1}{2}\Gamma(\frac{1}{2})\right]^2}{2} \\
&= \frac{\pi}{8}
\end{aligned}$$

3) Let's use integration by parts to evaluate $\int_0^x t^2 e^{-4t^2}\, dt.$

$$u = t, \quad dv = te^{-4t^2}\, dt \quad \Rightarrow \quad du = dt, \quad v = -\frac{e^{-4t^2}}{8}$$

$$\begin{aligned} \int_0^x t^2 e^{-4t^2}\, dt &= -\frac{te^{-4t^2}}{8}\Big|_0^x + \int_0^x \frac{e^{-4t^2}}{8} dt \\ &= -\frac{xe^{-4x^2}}{8} + \frac{1}{16}\int_0^{2x} e^{-u^2} du \end{aligned}$$

Using the definition of the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ we obtain

$$\int_0^x t^2 e^{-4t^2}\, dt = -\frac{xe^{-4x^2}}{8} + \frac{\sqrt{\pi}\ \mathrm{erf}(2x)}{32}$$

4) Adding and subtracting these equations, we obtain:

$$\frac{n}{x}\,F_n(x) + F_n'(x) \;=\; F_{n-1}(x)$$

$$\frac{n}{x}\,F_n(x) - F_n'(x) \;=\; F_{n+1}(x)$$

By changing indices and rearranging:

$$F_n(x) \;=\; \frac{n+1}{x}\,F_{n+1}(x) + F_{n+1}'(x)$$

$$F_{n+1}(x) \;=\; \frac{n}{x}\,F_n(x) - F_n'(x)$$

Insert $F_{n+1}(x)$ from second equation in the first:

$$F_n(x) = \frac{n+1}{x}\,\left[\frac{n}{x}\,F_n(x) - F_n'(x)\right] + \left[\frac{n}{x}\,F_n(x) - F_n'(x)\right]'$$

$$F_n(x) = \frac{n(n+1)}{x^2}\,F_n(x) - \frac{n+1}{x}\,F_n'(x) - \frac{n}{x^2}\,F_n(x) + \frac{n}{x}\,F_n'(x) - F_n''(x)$$

$$F_n(x) = \frac{n^2}{x^2}\,F_n(x) - \frac{1}{x}\,F_n'(x) - F_n''(x)$$

$$F_n''(x) + \frac{1}{x}\,F_n'(x) + F_n(x) - \frac{n^2}{x^2}\,F_n(x) = 0$$

$$x^2 F_n''(x) + x F_n'(x) + \left(x^2 - n^2\right) F_n(x) = 0$$

This is Bessel's ODE!

5) We know that the generating function for modified Bessel's function is

$$e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)}$$

in other words

$$e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x)\, t^n$$

Choosing $t = 1$ we obtain

$$e^x = \sum_{n=-\infty}^{\infty} I_n(x)$$

Note that $I_n(x) = I_{-n}(x)$:

$$e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$$

Similarly, choosing $t = -1$ we obtain

$$e^{-x} = \sum_{n=-\infty}^{\infty} I_n(x)(-1)^n$$

$$e^{-x} = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n I_n(x)$$