Çankaya University
Department of Mathematics
2020 - 2021 Spring Semester

# MATH 410 - Special Functions

# Second Midterm Examination

**1)** Find $P_7(x)$ (Legendre polynomial of order $7$) using any method you wish.

**2)** Show that

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}, \qquad n = 1, 2, 3, \ldots$$

where $P_n(x)$ is the Legendre polynomial of order $n$.

**3)** Using the generating function, show that

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \qquad n = 0, 1, 2, 3, \ldots$$

where $H_{2n}(x)$ is the Hermite polynomial of order $2n$.

**4)** Show that

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x), \qquad n = 1, 2, 3, \ldots$$

where $L_n(x)$ is the Laguerre polynomial of order $n$.
Hint: Use generating function.

**5)** Using the explicit series form, show that

$$L_n''(0) = \frac{1}{2}n(n-1)$$

where $L_n(x)$ is the Laguerre polynomial of order $n$.

# ANSWERS

**1)** We know that $P_0 = 1$ and $P_1 = x$ using the ODE. Let's use the recurrence relation

$$P_{n+1} = \frac{1}{n+1}\Big((2n+1)xP_n - nP_{n-1}\Big)$$

$$\begin{aligned} P_2 &= \frac{1}{2}\big(3xP_1 - P_0\big) \\ &= \frac{1}{2}\big(3x^2 - 1\big) \end{aligned}$$

$$\begin{aligned} P_3 &= \frac{1}{3}\big(5xP_2 - 2P_1\big) \\ &= \frac{1}{2}\big(5x^3 - 3x\big) \end{aligned}$$

$$\begin{aligned} P_4 &= \frac{1}{4}\big(7xP_3 - 3P_2\big) \\ &= \frac{1}{8}\big(35x^4 - 30x^2 + 3\big) \end{aligned}$$

$$\begin{aligned} P_5 &= \frac{1}{5}\big(9xP_4 - 4P_3\big) \\ &= \frac{1}{8}\big(63x^5 - 70x^3 + 15x\big) \end{aligned}$$

$$\begin{aligned} P_6 &= \frac{1}{6}\big(11xP_5 - 5P_4\big) \\ &= \frac{1}{16}\big(231x^6 - 315x^4 + 105x^2 - 5\big) \end{aligned}$$

$$\begin{aligned} P_7 &= \frac{1}{7}\big(13xP_6 - 6P_5\big) \\ &= \frac{1}{16}\big(429x^7 - 693x^5 + 315x^3 - 35x\big) \end{aligned}$$

**2)** Let's rewrite the recurrence relation as:

$$xP_n = \frac{1}{2n+1}\Big((n+1)P_{n+1} - nP_{n-1}\Big)$$

$$\begin{aligned}
\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx &= \int_{-1}^{1}\frac{1}{2n+1}\Big((n+1)P_{n+1} + nP_{n-1}\Big)P_{n-1}\,dx \\
&= \frac{n}{2n+1}\int_{-1}^{1} P_{n-1}^2\,dx \\
&= \frac{n}{2n+1}\frac{2}{2n-1} \\
&= \frac{2n}{4n^2-1}
\end{aligned}$$

**3)** The generating function for Hermite Polynomials:

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}$$

If $x = 0$ we obtain:

$$e^{-t^2} = \sum_{n=0}^{\infty} H_n(0)\,\frac{t^n}{n!}$$

$$\sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!} = \sum_{n=0}^{\infty} H_n(0)\,\frac{t^n}{n!}$$

Clearly, $H_n(0) = 0$ if $n$ is odd. Let's rewrite the right hand side using this:

$$\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{k!} = \sum_{n=0}^{\infty} H_{2n}(0)\,\frac{t^{2n}}{(2n)!}$$

In this equation, we can see that $n = k$ therefore:

$$H_{2n}(0) = \frac{(-1)^n(2n)!}{n!}$$

**4)** The generating function for Laguerre Polynomials:

$$\frac{1}{1-t}\,e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x)\,t^n$$

If we differentiate with respect to $t$, after some operations, we obtain

$$(n+1)L_{n+1} = (2n+1-x)L_n - nL_{n-1}$$

The derivative of this with respect to $x$ gives:

$$(n+1)L'_{n+1} = (2n+1-x)L'_n - L_n - nL'_{n-1}$$

$$(n+1)\Big(L'_{n+1} - L'_n\Big) = n\Big(L'_n - L'_{n-1}\Big) - xL'_n - L_n$$

If we differentiate the generating function with respect to $x$, after some operations, we obtain

$$L'_{n+1} = L'_n - L_n$$

Using the last two equations, we obtain

$$xL'_n = nL_n - nL_{n-1}$$

**5)** The explicit series form for Laguerre polynomials is:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}\,x^k = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2}\,x^k$$

$$L'_n(x) = \sum_{k=1}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2}\,kx^{k-1}$$

$$L''_n(x) = \sum_{k=2}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2}\,k(k-1)x^{k-2}$$

$L''_n(0)$ is the coefficient $k=2$:

$$L''_n(0) = \frac{(-1)^2 n!}{(n-2)!(2!)^2}\,2$$

$$L''_n(0) = \frac{n(n-1)}{2}$$